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THE *K*-GROUP MAXIMUM-FLOW NETWORK-INTERDICTION PROBLEM

by

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March 2000

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We study the *K*-group network-interdiction problem (KNIP) in which a "network user" attempts to maximize flow among $K \geq 3$ "node groups," while an "interdictor" interdicts (destroys) network arcs, using limited interdiction resources, to minimize this maximum flow. We develop two models to solve or approximately solve KNIP.

The multi-partition network-interdiction model (MPNIM) is an approximating model. It partitions the node set *N* into *K* different subsets, each containing one prespecified node group, and interdicts arcs using limited resources so that the total capacity of uninterdicted arcs crossing between subsets is minimized. The multi-commodity network-interdiction model (MCNIM) explicitly minimizes the maximum amount of flow that can potentially be moved among node groups using *K* single-commodity flow models connected by joint capacity constraints. It is a min-max model but is converted into an equivalent integer program MCNIM-IP.

Both MPNIM and MCNIM-IP are tested using four artificially constructed networks with up to 126 nodes, 333 arcs, *K*=5, and 20 interdictions allowed. Using a 333 MHz Pentium II personal computer, maximum solution times are 563.1 seconds for MPNIM but six of 16 MCNIM-IP problems cannot be solved in under 3,600 seconds.

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THE *K*-GROUP MAXIMUM-FLOW NETWORK-INTERDICTION PROBLEM

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ABSTRACT

We study the K-group network-interdiction problem (KNIP) in which a "network user" attempts to maximize flow among $K \geq 3$ "node groups," while an "interdictor" interdicts (destroys) network arcs, using limited interdiction resources, to minimize this maximum flow. We develop two models to solve or approximately solve KNIP.

The multi-partition network-interdiction model (MPNIM) is an approximating model. It partitions the node set N into K different subsets, each containing one prespecified node group, and interdicts arcs using limited resources so that the total capacity of uninterdicted arcs crossing between subsets is minimized. The multi-commodity network-interdiction model (MCNIM) explicitly minimizes the maximum amount of flow that can potentially be moved among node groups using K single-commodity flow models connected by joint capacity constraints. It is a min-max model but is converted into an equivalent integer program MCNIM-IP.

Both MPNIM and MCNIM-IP are tested using four artificially constructed networks with up to 126 nodes, 333 arcs, K=5, and 20 interdictions allowed. Using a 333 MHz Pentium II personal computer, maximum solution times are 563.1 seconds for MPNIM but six of 16 MCNIM-IP problems cannot be solved in under 3,600 seconds.

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I. INTRODUCTION

The study of network interdiction considers the problem of allocating firepower or other means to lessen the warfighting capabilities of an enemy through interdictions of his transportation and/or communications networks.

One well-studied interdiction problem, the "maximum-flow network-interdiction problem" (MFNIP), minimizes the maximum source-to-sink flow through a capacitated network by interdicting (attacking) network arcs with limited interdiction resources. This thesis investigates a generalization of MFNIP, the "*K*-group maximum-flow network-interdiction problem" (KNIP), and develops two models to solve or approximately solve this problem.

In this chapter, we provide background on network interdiction, introduce KNIP, and briefly explain the proposed models.

A. OVERVIEW

In MFNIP, a "network user" (enemy, adversary) attempts to maximize the amount of flow from a source node to a sink node in a capacitated network G while an "interdictor," using limited resources, strives to minimize this maximum flow by interdicting (breaking or stopping flow on) network arcs. Steinrauf (1991) and Wood (1993) use mathematical-programming methods to solve MFNIP. They develop a min-max formulation of MFNIP, the "maximum-flow network-interdiction model" (MFNIM), and then convert that into an integer-programming model (MFNIM-IP).

MFNIP is reasonable for situations in which flow moves from one or more source nodes to one or more sink nodes. The model assumes that (a) the interdictor has complete information about source and sink nodes in the network and (b) no node in the network

acts as a source and sink node simultaneously. Thus, MFNIP is sensible for simple scenarios, for instance, where the enemy must move materiel, equipment, or troops through a transportation network from one or more rear staging points (source nodes) up to one or more battlefield locations (sink nodes).

Not all wartime scenarios are so simple, however. For instance, an enemy may have K "regional forces" scattered around a theater of war and interconnected by a communications network G with known topology and link capacities. The interdictor would like to use limited interdiction resources to attack this network and minimize his adversary's inter-group communications capabilities.

We cannot use MFNIM to model the above problem. We assume G to be undirected and call this new problem the "K-group maximum-flow network-interdiction problem" (KNIP). In KNIP, the network user tries to maximize the amount of flow in G among three or more "special node groups," e.g., different regional forces, each represented as a group of nodes; the interdictor uses limited resources to attack and minimize that maximum flow. As the interdictor, the assumption that the adversary will maximize flow among the special node groups allows us to minimize the worst-case functionality (for us) of the enemy's system. This provides an upper bound on the true quantity of materiel or message traffic that will be moved through the network, and this is probably the best we can do without knowing the value or purpose of particular flows.

KNIP may arise in situations where the interdictor has a rough idea about the locations of the enemy's force groups (or supply points, demand points, etc.,) but is unsure of exactly how much, and between which force groups, materiel or messages will be transferred. To solve KNIP, we develop two models, namely the "multi-partition

network-interdiction model" (MPNIM) and the "multi-commodity network-interdiction model" (MCNIM). In both MCNIM and MPNIM, we assume that (a) the network $G=(N,A)$ is capacitated and undirected and (b) K disjoint "special node groups" $N'_1,\ldots,N'_K$ are prespecified where each N'_k represents a set of source and/or sink nodes. Furthermore, $\left|\bigcup_k N'_k\right| << |N|$ is assumed, although this assumption does not materially affect any of the solution techniques.

MCNIM is a *direct* model for KNIP, i.e., it explicitly minimizes the maximum amount of flow that can be moved among the node groups $N'_1,\ldots,N'_K$. MCNIM models the potential movement of enemy supplies or messages using K single-commodity maximum-flow models connected by joint capacity constraints: For each k', $N'_{k'}$ is treated as a set of source nodes and $\bigcup_{k\neq k'} N'_k$ is treated as a set of sink nodes. The interdictor destroys arcs to minimize the maximum amount of flow that can be moved in this K-commodity model.

MCNIM is a complicated model that may be difficult to solve. Therefore, we first describe and solve a smaller approximating model MPNIM. MPNIM partitions the node set N into K subsets $N_1,\ldots,N_K$, with $N'_k \subseteq N_1,\ldots,N'_K \subseteq N_K$, and interdicts certain arcs connecting the subsets N_k while observing constraints on interdiction resources. The objective is to minimize the total capacity of the uninterdicted arcs crossing between the various subsets.

MPNIM is simpler than MCNIM and may be easier to solve. Furthermore, we show that it can be modified to completely isolate K node groups, if desired. However,

MPNIM is an approximating model because (a) it does not explicitly minimize flow among the node groups and (b) it does not enforce the sensible requirement that the nodes in each identified subset be contiguous.

KNIP is equivalent to MFNIP when the number of disjoint node groups K=2 and the network G=(N,A) is undirected. Thus, MFNIP may be regarded as a special case of KNIP.

B. LITERATURE SEARCH

Several network-interdiction problems have been studied in the literature, but most research has dealt with MFNIP. MFNIP was originally motivated by efforts to destroy enemy supply lines during the Vietnam War. The problem has been studied by Wollmer (1964, 1970), Durbin (1966), McMasters and Mustin (1970), Helmbold (1971), Ghare, Montgomery, and Turner (1971), Lubore, Ratliff, and Sicilia (1971, 1975), Steinrauf (1991), and Wood (1993). MFNIM, which is a starting point for this thesis, appears in the last two works. Cormican (1995) shows how to solve MFNIM using Benders decomposition and Cormican, Morton and Wood (1995) solve a stochastic version of MFNIM with uncertainty in interdiction successes and/or arc capacities.

Another category of network-interdiction models is also well-studied, namely "maximizing the shortest path" (MXSP). In MXSP, a network user wishes to traverse a shortest path from a specified node s to a specified node t in a directed (or undirected) network G=(N,A) whose arc lengths $c_{ij} \geq 0$ are known. An interdictor attempts to interdict (destroy or lengthen) arcs, using limited interdiction assets, to maximize the length of all shortest-paths. Contributors in this area are Fulkerson and Harding (1977),

Golden (1978), Israeli (1999), Israeli and Wood (1999) and Wevley (1999). (Wevley's model is actually a variant on MXSP.)

Steinrauf (1991) studies a network-interdiction model that isolates a node or a set of nodes in a drug-interdiction scenario. His model destroys arcs to isolate the maximum number of nodes around a "central node" in an attempt to maximize the chance of isolating a drug-supply node that is believed to lie near the central node. The model identifies a set of arcs to interdict and the set of isolated nodes.

Reed (1994) devises an integer-programming model to maximize the longest path in a PERT network through interdiction. The PERT network represents significant tasks or "activities" of a project together with precedence relations between activities and the nominal time required to complete each activity. Interdictions lengthen the time required to complete activities and can therefore be used to delay project completion. Reed constructs an interdiction model to delay the proliferation of nuclear weapons.

Wollmer (1970) and Washburn and Wood (1994) develop game-theoretic network-interdiction models. However, the purpose of these models is quite different than those mentioned above. In particular, these authors try to determine optimal arc-inspection strategies to detect an evader moving through a network secretly. These models are not related to this thesis.

C. OUTLINE OF THESIS

The remainder of this thesis is organized as follows. Chapter II gives essential definitions and notation together with a detailed background on MFNIM. In Chapters III and IV, we develop MPNIM and MCNIM, respectively. We give the computational

results regarding both models in these chapters. Chapter V provides conclusions and recommendations for further research.

II. PRELIMINARIES

In this chapter, we give essential definitions and notation together with a detailed derivation of MFNIM. Most definitions follow Wood (1993) and Ahuja, Magnanti, and Orlin (1993).

Although KNIP is defined on an undirected network, much of the related theory is based on directed networks. Therefore, we use those networks as a starting point and discuss undirected networks later.

A. DEFINITIONS AND NOTATION

$G=(N,A)$ denotes a directed network with node set N and arc set A. An arc is an ordered pair (i,j) with $i, j \in N$, and $i \neq j$. For an arc (i,j), i is the "tail node" from which the arc originates, and j is the "head node" at which the arc terminates. An arc (i,j) is "incident from i" and "incident to j." FS(i) (forward star of node i) represents the set of arcs incident from node i and RS(i) (reverse star of node i) represents the set of arcs incident to node i.

In this thesis, we assume a single type of interdiction resource. R total units of resource are available to the interdictor, and r_{ij} units of resource are required to interdict arc (i,j). The r_{ij} may be assumed to be small, positive integers.

B. NETWORK MAXIMUM-FLOW MODELS

1. Standard Maximum-Flow Model

In MFNIM, the network user is assumed to solve a maximum-flow model after observing the effects of interdictions. In this section, we define the "standard" maximum-flow model (MFM) (e.g., Ahuja, Magnanti, and Orlin 1993, p. 168) that

determines the maximum quantity of a single commodity that can be moved through a capacitated network from a source node s to a sink node t.

We consider a capacitated, directed network $G=(N,A)$ with a nonnegative capacity $u_{ij} < \infty$ associated with each arc $(i,j) \in A$. Let $U = \max_{(i,j)\in A} u_{ij}$. To define the "maximum-flow problem," we distinguish two special nodes in G, a "source node" s and a "sink node" t, $t \neq s$. Maximizing flow from s to t is the same as maximizing flow along an extra "return arc" (t,s) added to G.

Maximum-Flow Model (MFM)

Indices:

$i, j \in N$ nodes in an directed network $G = (N, A)$. Includes two special nodes, the source s and the sink t .

$(i, j) \in A$ directed arcs in the network $G = (N, A)$

Data:

u_{ij} nominal capacity of arc (i,j)

Decision Variables:

y_{ij} amount of flow on arc (i,j)

Formulation:

$$\max_{y} \quad y_{ts} \qquad : \text{dual variables}$$

$$\text{s.t.} \quad \sum_{(s,j)\in FS(s)} y_{sj} - \sum_{(j,s)\in RS(s)} y_{js} - y_{ts} = 0 \qquad : \alpha_s \qquad (1)$$

$$\sum_{(i,j)\in FS(i)} y_{ij} - \sum_{(j,i)\in RS(i)} y_{ji} = 0 \qquad \forall i \in N - s - t \qquad : \alpha_i \qquad (2)$$

$$\sum_{(t,j)\in FS(t)} y_{tj} - \sum_{(j,i)RS(t)} y_{jt} + y_{ts} = 0 \qquad : \alpha_t \tag{3}$$

$$0 \le y_{ij} \le u_{ij} \qquad \forall (i,j) \in A \qquad : \theta_{ij} \tag{4}$$

The quantity y_{ij} is the flow of the commodity from node i to node j on directed arc $(i,j) \in A$, and y_{ts} is the flow from sink node t to source node s on the artificial return arc (t,s). The "flow-balance constraints" (1), (2) and (3) require that the flow arriving at a node equal the flow leaving the node. Capacity constraints (4) require that flow on each arc be non-negative and not exceed the arc's capacity.

2. Simplified Maximum-Flow Model

An equivalent and simpler formulation of the maximum-flow model can be obtained if flow entering the sink node t or flow leaving the source node s is maximized. This formulation, which will simplify our later models, is:

<u>Simplified Maximum-Flow Model (SMFM)</u>

$$\max_{\mathbf{y}} \sum_{(s,j)\in FS(s)} y_{sj} \tag{5}$$

$$\text{s.t.} \quad \sum_{(i,j)\in FS(i)} y_{ij} - \sum_{(j,i)\in RS(i)} y_{ji} = 0 \qquad \forall i \in N - s - t \tag{6}$$

$$y_{is} \equiv 0 \qquad \forall (i,s) \in RS(s) \tag{7}$$

$$y_{tj} \equiv 0 \qquad \forall (t,j) \in FS(t) \tag{8}$$

$$0 \le y_{ij} \le u_{ij} \qquad \forall (i,j) \in A \tag{9}$$

The objective (5) is to maximize flow entering the sink node t, although it can be replaced by $\max_{\mathbf{y}} \sum_{(j,i)\in RS(t)} y_{it}$ which maximizes flow leaving the source node s. Constraints (6) are the standard flow-balance constraints for nodes other than s and t. Constraints (7)

and (8) set the flow on arcs that terminate at s and leave t to zero, respectively. Capacity constraints (9) are as before.

3. Cuts and the Dual of the Maximum-Flow Model

A "cut" (N_s, N_t) is a partition of the node set N into two subsets N_s and N_t such that $s \in N_s$ and $t \in N_t$. Each cut defines a set of arcs that have one endpoint in N_s and the other endpoint in N_t. With respect to that cut, an arc (i, j) is a "forward arc" if $i \in N_s$ and $j \in N_t$; otherwise it is "backward arc." The "capacity of a cut" (N_s, N_t) is $\sum_{(i,j)\in A | i\in N_s, j\in N_t} u_{ij}$, i.e., the capacity of the cut is the sum of arc capacities for the forward arcs crossing the cut. A "minimum cut" is a cut whose capacity is minimum among all possible cuts in the network.

By the well-known maximum-flow minimum-cut theorem (Ford and Fulkerson 1956), the maximum flow equals the capacity of a minimum cut. A minimum cut can be found directly by solving the dual of the maximum-flow problem DMFM (e.g., Wood 1993). DMFM, which we will show next, is important for formulating MFNIM-IP.

The dual variables of the maximum-flow model, α_i and θ_{ij}, have already been indicated in MFM. When we find an optimal solution to the maximum-flow problem, we also find an optimal solution to the minimum-cut problem through those dual variables.

<u>Dual of the Maximum-Flow Model (DMFM)</u>

Indices:

$i, j \in N$	nodes in an directed network $G = (N, A)$. Includes two special nodes, the source s and the sink t
$(i, j) \in A$	directed arcs in the network $G = (N, A)$

Data:

u_{ij} nominal capacity of arc (i,j)

Decision Variables:

α_i dual variables associated with flow-balance constraints (1), (2) and (3) from MFM

θ_{ij} dual variables associated with capacity constraints (4) in MFM

Formulation:

$$\min_{\alpha,\beta} \quad \sum_{(i,j)\in A} u_{ij}\theta_{ij}$$

$$\text{s.t.} \quad \alpha_i - \alpha_j + \theta_{ij} \geq 0 \qquad \forall (i,j) \in A \tag{10}$$

$$\alpha_t - \alpha_s \geq 1 \tag{11}$$

$$\theta_{ij} \geq 0 \qquad \forall (i,j) \in A$$

In fact, the dual variables in DMFM may be assumed to be binary (e.g., Wood 1993). Let (N_s, N_t) correspond to a minimum cut in $G=(N,A)$ and let $\alpha_i = 1$ for $\forall i \in N_t$, $\alpha_i = 0$ for $\forall i \in N_s$, $\theta_{ij} = 1$ for all arcs (i,j) which are forward arcs in the cut, and $\theta_{ij} = 0$ for all other arcs. Constraint (11) is obviously satisfied by the assignment of the variables. We can also see that constraints (10) are satisfied by checking against the four classes of arcs (i,j): (a) $i \in N_s, j \in N_s$, (b) $i \in N_s, j \in N_t$, (c) $i \in N_t, j \in N_s$, and (d) $i \in N_t, j \in N_t$. Thus, the above solution is feasible. Furthermore, because the objective function of DMFM equals the capacity of the minimum cut (N_s, N_t), it follows

from maximum-flow minimum-cut theorem and linear-programming duality that the solution is optimal. Wood (1993) uses this result to convert MFNIM into MFNIM-IP.

4. Undirected Networks

$G=(N,A)$ may also denote an undirected network. Undirected networks are important to us because (a) many real-world networks such as road and telecommunications networks are essentially undirected, (b) KNIP will be defined on undirected networks and (c) although MFNIM was originally defined for directed networks, Wood (1993) shows how to extend it to undirected networks.

An undirected network is defined in the same manner as a directed network except that arcs are unordered pairs of distinct nodes. In an undirected network, we can refer to an arc joining the node pair i and j as either (i,j) or (j,i). The arc (i,j) is said to be "incident to" both nodes i and j. $A(i)$ denotes the set of arcs incident to node i.

Flow on an undirected arc (i,j) can move from i to j, which will be represented by y_{ij}, or it can move from j to i, which will be represented by y_{ji}. The total flow (i.e., from node i to node j plus from node j to node i) on an undirected arc (i,j) has an upper bound u_{ij}. That is, the maximum-flow model for an undirected network G has capacity constraints $y_{ij} + y_{ji} \le u_{ij} \quad \forall (i,j) \in A$. So, a maximum-flow model for an undirected network $G=(N,A)$ is:

Maximum-Flow Model for an Undirected Network

$$\max_{\mathbf{y}} \quad y_{ts}$$

$$\text{s.t.} \quad \sum_{j:(s,j)\in A \cup (j,s)\in A} y_{sj} - \sum_{j:(s,j)\in A \cup (j,s)\in A} y_{js} - y_{ts} = 0$$

$$\sum_{j:(i,j)\in A \cup (j,i)\in A} y_{ij} - \sum_{j:(i,j)\in A \cup (j,i)\in A} y_{ji} = 0 \qquad \forall i \in N - s - t$$

$$\sum_{j:(t,j)\in A \cup (j,t)\in A} y_{tj} - \sum_{j:(t,j)\in A \cup (j,t)\in A} y_{js} - y_{ts} = 0$$

$$y_{ij} + y_{ji} \le u_{ij} \qquad \forall (i,j) \in A$$

$$y_{ij}, y_{ji} \ge 0 \qquad \forall (i,j) \in A$$

The above model is equivalent to a maximum-flow model on a directed network $G^+ = (N, A^+)$ where A^+ denotes the set of anti-parallel directed arcs derived from the set of undirected arcs A so that $|A^+| = 2|A|$.

In an *s-t* maximum-flow model for an undirected network *G*, adding simple upper bounds $y_{ij} \le u_{ij}$ and $y_{ji} \le u_{ij}$ instead of capacity constraints $y_{ij} + y_{ji} \le u_{ij}$, suffices because there always exists an optimal flow with $y_{ij} = 0$ or $y_{ji} = 0$ (e.g., Ahuja, Magnanti, and Orlin 1993, p. 39). In MCNIM, extensions of the constraints $y_{ij} + y_{ji} \le u_{ij}$ will be required; simple upper bounds will not suffice.

C. MAXIMUM-FLOW NETWORK-INTERDICTION

1. Maximum-Flow Network-Interdiction Model

MFNIP can be formalized in a min-max flow-based model. The network user attempts to maximize the flow across the network, while the interdictor simultaneously strives to minimize this maximum flow. The network interdictor's activities are limited

by a resource constraint. We call the resulting model the "maximum-flow network-interdiction model" (MFNIM).

Maximum-Flow Network-Interdiction Model (MFNIM)

Indices:

$i, j \in N$ nodes in an directed network $G = (N, A)$. Includes two special nodes, the source s and the sink t

$(i, j) \in A$ directed arcs in the network $G = (N, A)$

Data:

u_{ij} nominal capacity of arc (i,j)

r_{ij} interdiction resource required to interdict (break) arc (i,j)

R total interdiction resource

Decision Variables:

Network User's Decision Variables:

y_{ij} amount of flow on arc (i,j)

Interdictor's Decision Variables:

x_{ij} 1 if arc (i,j) is interdicted; 0 otherwise

Formulation:

$$\min_{\mathbf{x} \in X} \max_{\mathbf{y}} \quad y_{ts} \tag{12}$$

$$\text{s.t.} \quad \sum_{(s,j) \in FS(s)} y_{sj} - \sum_{(j,s) \in RS(s)} y_{js} - y_{ts} = 0 \tag{13}$$

$$\sum_{(i,j) \in FS(i)} y_{ij} - \sum_{(j,i) \in RS(i)} y_{ji} = 0 \qquad \forall i \in N - \{s,t\} \tag{14}$$

$$\sum_{(t,j)\in FS(t)} y_{tj} - \sum_{(j,t)\in RS(t)} y_{jt} + y_{ts} = 0 \tag{15}$$

$$0 \le y_{ij} \le u_{ij}(1 - x_{ij}) \qquad \forall (i,j) \in A \tag{16}$$

$$\text{where } X = \{\mathbf{x} \in \{0,1\}^{|A|} : \sum_{(i,j)\in A} r_{ij} x_{ij} \le R\} \tag{17}$$

The objective (12) is to minimize the maximum flow. Constraints (13), (14) and (15) are just flow-balance constraints from MFM. The capacity constraints (16) restrict the amount of flow on each arc to the arc's nominal capacity if the arc is not interdicted, or to zero if the arc is interdicted. Constraint (17) limits the expenditure of interdiction resource.

2. An Equivalent Integer Program

For a fixed interdiction decision, note that the inner maximization of the MFNIM is just a maximum-flow model. Wood (1993) takes the dual of the inner maximum-flow model and linearizes the resulting non-linear objective function to obtain an equivalent (linear) integer program. We refer to new model as "MFNIM-IP" and describe it next.

Maximum-Flow Network-Interdiction As An Integer Program (MFNIM-IP)

Indices:

$i, j \in N$ nodes in an directed network $G = (N, A)$. Includes two special nodes, the source s and the sink t

$(i, j) \in A$ directed arcs in the network $G = (N, A)$

Data:

u_{ij} nominal capacity of arc (i,j)

r_{ij} interdiction resource required to interdict (break) arc (i,j)

R total interdiction resource

Decision Variables:

x_{ij} 1 if arc (i,j) is interdicted; 0 otherwise

α_i for some cut (N_s, N_t), 1 if $i \in N_s$; 0 if $i \in N_t$

β_{ij} 1 if arc (i,j) is a forward arc of the cut (N_s, N_t) and is not interdicted; 0 otherwise

Formulation:

$$\min_{\alpha,\beta,x} \quad \sum_{(i,j)\in A} u_{ij}\beta_{ij}$$

$$\text{s.t} \quad \alpha_i - \alpha_j + x_{ij} + \beta_{ij} \geq 0 \qquad \forall (i,j) \in A$$

$$\alpha_t - \alpha_s \geq 1$$

$$\sum_{(i,j)\in A} r_{ij} x_{ij} \leq R$$

$$\beta_{ij}, x_{ij} \in \{0,1\} \qquad \forall (i,j) \in A$$

$$\alpha_i \in \{0,1\} \qquad \forall i \in N$$

Note that this model resembles DMFM but with θ_{ij} replaced by $\beta_{ij} + x_{ij}$. MFNIM identifies a cut where the variables α_i have the same meaning as in the DMFM. x_{ij} and β_{ij} represent interdiction decisions and can be explained as follows: For a forward arc (i,j) crossing the cut, $\alpha_i - \alpha_j = -1$ so $\beta_{ij} + x_{ij} = 1$ is required. So, either $x_{ij} = 1$, indicating that this arc is interdicted, or $\beta_{ij} = 1$, indicating that this arc is not interdicted and forms part of the minimum cut after interdiction. $x_{ij} = \beta_{ij} = 0$ indicates

that arc (i,j) is neither interdicted nor part of the (identified) minimum cut after interdiction.

D. AN EQUIVALENT FORMULATION OF MFNIM

The original derivation of MFNIM-IP from MFNIM is complicated (Wood 1993). However, a simpler derivation is possible. This derivation will be useful later and is provided for reference, next.

We know that the objective function (12) of MFNIM is a concave function in the x_{ij} and so it is clear that MFNIM is a difficult, non-convex minimization problem even if the x_{ij} are continuous. The problem, however, can be "convexified" by moving the variables x_{ij} into the objective of the inner maximization (Cormican, Morton and Wood 1998). In the following, we give this formulation and explain why this technique works. We call the new model the "convexified maximum-flow network-interdiction model" (MFNIMc).

<u>Convexified Maximum-Flow Network-Interdiction Model (MFNIMc)</u>

$$\min_{\mathbf{x}\in X} \max_{\mathbf{y}} \quad y_{ts} - \sum_{(i,j)\in A} y_{ij} x_{ij} \tag{18}$$

$$\text{s.t.} \quad \sum_{(s,j)\in FS(s)} y_{sj} - \sum_{(j,s)\in RS(s)} y_{js} - y_{ts} = 0 \tag{19}$$

$$\sum_{(i,j)\in FS(i)} y_{ij} - \sum_{(j,i)\in RS(i)} y_{ji} = 0 \qquad \forall i \in N - s - t \tag{20}$$

$$\sum_{(t,j)\in FS(t)} y_{tj} - \sum_{(j,t)\in RS(t)} y_{jt} + y_{ts} = 0 \tag{21}$$

$$0 \le y_{ij} \le u_{ij} \qquad \forall (i,j) \in A \tag{22}$$

where $X = \{\mathbf{x} \in \{0,1\}^{|A|} : \sum_{(i,j)\in A} r_{ij} x_{ij} \le R\}$.

MFNIM and MFNIMc are essentially equivalent because the inner maximization in MFNIMc is essentially equivalent to the inner maximization in MFNIM. The inner objective in MFNIMc maximizes the maximum flow through the network less flow on interdicted arcs. Because the value of a unit of flow on an arc in a maximum-flow problem is at most one, there can be no benefit to the interdictor of having flow on interdicted arcs. Therefore, if $x_{ij} = 1$, we may assume that, as in MFNIM, $y_{ij} = 0$. (It is possible in MFNIMc that $y_{ij} > 0$ even if $x_{ij} = 0$, but in this case there always exists an alternative optimal solution with $y_{ij} = 0$.)

MFNIMc allows us to obtain MFNIM-IP directly. To do this, fix the integer variables x_{ij}, take the dual of the inner maximization and then release the x_{ij}.

Building on the material discussed in this chapter, we introduce MPNIM in Chapter III and MCNIM in Chapter IV, respectively.

III. MULTI-PARTITION NETWORK-INTERDICTON MODEL

In this chapter, we present the "multi-partition network-interdiction model" (MPNIM) that solves KNIP approximately.

A. OVERVIEW

Recall from Chapter I that MFNIM cannot be used to solve KNIP. However, we can use some of the ideas behind MFNIM to solve KNIP approximately.

MFNIM is based on the maximum-flow minimum-cut theorem (Ford and Fulkerson, 1956). Wood (1993) shows that MFNIM identifies a cut and breaks certain arcs in that cut so as to minimize remaining cut capacity. In other words, MFNIM partitions the node set N into two subsets N_s and N_t, with N_s containing a specified source node s (or source nodes) and N_t containing a specified sink node t (or sink nodes), and interdicts arcs that cross between these subsets, so that the flow between the subsets N_s and N_t is minimized. We use this node-partitioning idea to develop MPNIM.

We assume that the network $G=(N,A)$ is undirected and that K disjoint "special node subsets" $N'_1,...,N'_K$ are prespecified; each N'_k represents a set of source and/or sink nodes whose identities can be obtained through intelligence reports. Instead of trying to minimize the flow among the $N'_k,...,N'_K$ directly (which results in MCNIM), MPNIM will partition N into K disjoint subsets $N_1,...,N_K$, with $N'_1 \subseteq N_1,...,N'_K \subseteq N_K$, and interdict certain arcs connecting the subsets N_k while observing constraints on interdiction resources. The objective is to minimize the total capacity of the uninterdicted arcs crossing between the subsets N_k.

This model is appealing because (a) we expect there to be a strong correlation between the minimized "total inter-subset capacity" and actual flow, (b) the optimal objective value of MPNIM clearly provides an upper bound on inter-subset flows, i.e., on the optimal solution to KNIP, and (c) the model reduces to MFNIM when K=2.

We next give the formulation for MPNIM and explain the formulation in detail. We then extend the formulation to isolate the special node groups completely.

B. INTERDICTION MODEL MPNIM

MPNIM can be formulated as follows:

Multi-Partition Network-Interdiction Model (MPNIM)

Indices:

$i, j \in N$ nodes in an undirected network $G=(N,A)$

$(i, j) \in A$ undirected arcs in the network $G=(N,A)$

N'_k nodes that are preassigned to node subset N_k, $k = 1,\ldots, K$.

$N'_k \cap N'_{k'} = \varnothing \quad \forall k \neq k'$, $N'_k \neq \varnothing$ for any k

Data:

u_{ij} nominal capacity of arc (i,j)

r_{ij} interdiction resource required to interdict (break) arc (i,j)

R total interdiction resource

Decision Variables:

x_{ij} if arc (i,j) crosses between two different subsets and is interdicted;

0 otherwise

α_{ik} 1 if node i is assigned to N_k; 0 otherwise

β_{ij} 1 if arc (i,j) crosses between two different subsets and is not interdicted;

0 otherwise

Formulation:

$$\min_{\mathbf{x},\boldsymbol{\beta},\boldsymbol{\alpha}} \sum_{(i,j)\in A} u_{ij}\beta_{ij} \tag{23}$$

s.t.

$$\sum_k \alpha_{ik} = 1 \qquad i \in N \tag{24}$$

$$x_{ij} + \beta_{ij} + \alpha_{ik} - \alpha_{jk} \geq 0 \qquad k = 1,\ldots,K,\ (i,j) \in A \tag{25}$$

$$x_{ij} + \beta_{ij} + \alpha_{jk} - \alpha_{ik} \geq 0 \qquad k = 1,\ldots,K,\ (i,j) \in A \tag{26}$$

$$x_{ij} + \beta_{ij} \leq 1 \qquad (i,j) \in A \tag{27}$$

$$\sum_{(i,j)\in A} r_{ij} x_{ij} \leq R \tag{28}$$

$$\beta_{ij}, x_{ij} \in \{0,1\} \qquad (i,j) \in A$$

$$\alpha_{ik} \in \{0,1\} \qquad k = 1,\ldots,K,\ i \in N$$

$$\alpha_{ik} \equiv 1 \qquad k = 1,\ldots,K,\ i \in N'_k \tag{29}$$

$$\alpha_{ik'} \equiv 0 \qquad \forall k', i \in N'_k, k \neq k' \tag{30}$$

The objective (23) is to minimize the sum of the capacities on uninterdicted arcs crossing between different node subsets. Constraints (24) require each node i to belong to exactly one subset N_k. Constraints (25) and (26) enforce a partitioning of the nodes and determine whether an arc crosses between two subsets:

1. If $i, j \in N_k$, then $\alpha_{ik} - \alpha_{jk} = 0$ and $\alpha_{jk} - \alpha_{ik} = 0$, which allows $x_{ij} = 0$ and $\beta_{ij} = 0$. $x_{ij} = 1$ and/or $\beta_{ij} = 1$ are also feasible to constraints (25) and (26) in this case, but we may assume that both variables are 0 because (a) $\beta_{ij} = 0$

contributes less to the objective function than does $\beta_{ij} = 1$, and (b) $x_{ij} = 0$ consumes less resource than does $x_{ij} = 1$. (Alternate optimal solutions with $x_{ij} = 1$ are possible if there is excess resource.)

2. If $i \in N_k$ and $j \in N_{k'}, k \neq k'$, then $x_{ij} + \beta_{ij} = 1$ is required to maintain feasibility. So, either $x_{ij} = 1$, indicating that arc (*i,j*) is interdicted or $\beta_{ij} = 1$, indicating that this arc is not interdicted and contributes to the inter-subset capacity after interdiction.

MPNIM classifies the arcs in the network into three groups: (a) Arcs that cross between subsets and are interdicted, (b) arcs that cross between subsets and are not interdicted, and (c) arcs that do not cross between subsets. Constraints (27) together with (25) and (26) ensure that each arc is in one of these three groups. Constraint (28) ensures that total interdiction resource consumed does not exceed total resource available. More complicated resource constraints involving the x_{ij} are certainly possible and do not materially affect the model. Constraints (29) set $\alpha_{ik} = 1$ if node *i* is preassigned to node subset N_k, i.e., $i \in N'_k$, and constraints (30) set $\alpha_{ik'} = 0$ to zero if $i \in N'_k$ and $k \neq k'$.

MPNIM is an approximation of MFNIM for two reasons. First, the model does not explicitly minimize flow among the special node groups N'_k. Second, MPNIM does not enforce a sensible requirement that the nodes in each identified subset be contiguous. However, the solution to the proposed model may be a reasonable approximation and should provide insight into a model that explicitly minimizes inter-subset flows.

C. EXTENDING MPNIM TO ISOLATE NODE GROUPS

We can extend MPNIM to isolate K special node groups completely. The simplest way to do this is (a) fix all $\beta_{ij} \equiv 0$, and (b) change the objective to $\min \sum_{(i,j)\in A} r_{ij} x_{ij}$, i.e., to minimize total interdiction resource consumption.

D. COMPUTATIONAL RESULTS

We have tested MPNIM using four artificially constructed networks assuming that $r_{ij} = 1$ for all arcs (i,j). Test networks are $n_1 \times n_2$ grid networks as shown in Figure 1, where n_1 is the number of nodes in the horizontal axis and n_2 is the number of nodes in the vertical axis. The arc capacities u_{ij} are randomly drawn from the discrete uniform distribution on [13,99]. The model is formulated in GAMS (Brooke, Kendrick, Meeraus and Raman 1997) and run on a 333 MHz Pentium II PC using the solvers CPLEX 6.5 and XA (GAMS Development Corporation 1997). Solution times for MPNIM are mostly better using XA; however, we present CPLEX's results because solution times for MCNIM-IP, which is harder to solve than MPNIM, are better using CPLEX. We use a relative optimality criterion (OptCR) of 1%. That is, the solver stops when the integrality gap $(|BP - BF|)/(1 + |BP|) < \text{OptCR}$ where BF is the objective function value of the current best integer solution and BP is the best possible integer solution (GAMS Development Corporation 1997). Table 1 gives model statistics and run times for several values of R and K for each of the four networks. Figure 1 displays one of the solutions.

It is not hard to find (by hand) a feasible multi-commodity flow in Figure 1 in which all non-interdicted, inter-subset arcs are capacitated. Thus, this interdiction plan is optimal for MCNIM-IP as well as for MPNIM.

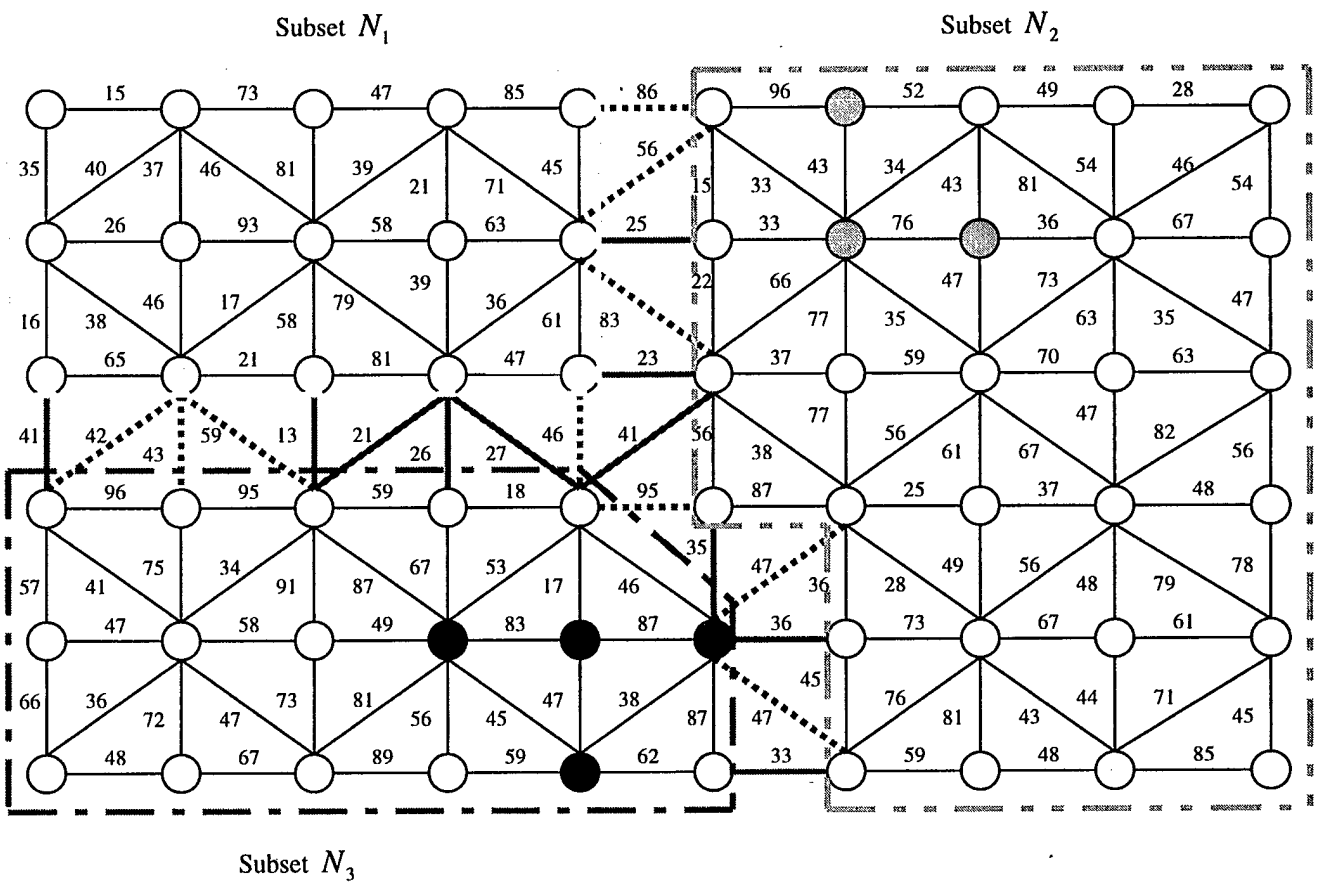
Network $G=(N,A)$	n_1	n_2	$\|N\|$	$\|A\|$	Total Resource R	Number of Subsets K	Optimal Obj. Value	Run Time (seconds)
1	7	4	28	63	9	3	16	1.6
					11	3	0	1.8
					6	4	334	1.7
					11	4	144	1.8
2	10	6	60	149	10	3	321	4.2
					11	4	427	20.3
					16	4	222	18.2
					25	4	0	25.6
3	14	7	98	263	11	3	234	4.2
					20	3	0	39.5
					16	4	313	6.4
					20	4	149	5.9
4	14	9	126	333	11	4	863	189.2
					20	4	426	160.9
					11	5	1414	518.5
					20	5	924	563.1

Table 1. Model statistics and run times for MPNIM using a relative optimality criterion of 1%. Several different values of R and K are used with four different networks. For instance, the problem for Network 2, which has 60 nodes and 149 arcs, when R=10 and K=3 is solved in 4.2 seconds with an optimal objective value of 321.

We have also tested MPNIM to isolate K node groups completely, extending the model as described in Section C. We use the same test networks, computer, solver and optimality criterion. Table 2 gives model statistics and run times for different values of K for each of the networks. Note that the optimal objective value is the amount of the interdiction resource required to isolate the node groups.

Network $G=(N,A)$	n_1	n_2	$\|N\|$	$\|A\|$	Number of Subsets K	Optimal Obj. Value	Run times (seconds)
1	7	4	28	63	3	10	1.5
					4	18	2.1
2	10	6	60	149	3	20	2.8
					4	23	10.6
3	14	7	98	263	3	19	3.6
					4	26	3.7
4	14	9	126	333	4	34	186.5
					5	46	484.4

Table 2. Model statistics and run times for MPNIM when used to isolate K node groups completely. A relative optimality criterion of 1% is used. Several different values of K are used with four different networks. For instance, the problem for Network 2 when K=3 is solved in 2.8 seconds with an optimal objective value of 20. The optimal objective value is actually the amount of interdiction resource needed to isolate 3 prespecified node groups completely.



Arcs that cross between two subsets and are interdicted. (There are 10 of these arcs.)

Arcs that cross between two subsets and are not interdicted. (There are 11 of these arcs.)

Arcs that do not cross between subsets and are in a subset. (There are 128 of these arcs.)

Figure 1. Solution of MPNIM for Network 2 when R=10, K=3 and $r_{ij} = 1 \quad \forall\, (i, j) \in A$. Numbers next to arcs are capacities. Each shade of darkened node indicates a different subset N'_k. The optimal objective value is 321.

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IV. MULTI-COMMODITY NETWORK-INTERDICTION MODEL

In Chapter III, we developed MPNIM to solve KNIP approximately. In this chapter, we derive two models to solve KNIP exactly. These models are the "multi-commodity network-interdiction model" (MCNIM) and an equivalent formulation of MCNIM, the "convexified multi-commodity network-interdiction model" (MCNIMc). Through MCNIMc, we also obtain an equivalent integer program for MCNIM, denoted "MCNIM-IP."

A. OVERVIEW

We assume that our adversary will try to maximize flow among K disjoint and prespecified node groups $N'_1,\ldots,N'_K$ in a capacitated, undirected network $G=(N,A)$. We model this using K single-commodity maximum-flow models connected by joint capacity constraints. (This is actually a multi-commodity maximum-flow model.) Each single-commodity maximum-flow model indexed by k treats nodes in N'_k as source nodes and nodes in $\bigcup_{k'\neq k} N'_{k'}$ as sink nodes; the adversary's objective is to maximize the sum of all single-commodity flows.

Having modeled the enemy's activities, the interdictor's problem is to minimize the maximum amount of flow among the N'_k using limited interdiction resources. As in MFNIM, this interdiction problem can be modeled using two different min-max flow-based formulations: (a) The basic formulation, MCNIM, sets the capacity of an arc to zero if interdicted and (b) the convexified version of the basic formulation, MCNIMc, subtracts flow on interdicted arcs from the standard maximum-flow objective.

Our assumption that the adversary will maximize flow allows us to model the worst-case functionality (for the interdictor) of the adversary's system. The purpose of the adversary is really to move the "right amount of commodities" between the "right node groups." By minimizing maximum flow, we minimize an upper bound on the true amount of commodities, measured in common units, that can be transferred. This is the best we can do without additional information on the value of flows in the network.

For modeling purposes, we define $N' = \bigcup_k N'_k$ to be "special nodes" and $N - N'$ to be "non-special nodes." Our formulations assume that all special nodes can be both source and sink nodes; however, the formulations can easily be adapted to situations in which some special nodes are only source or only sink nodes.

We now explain MCNIM, MCNIMc and MCNIM-IP in detail.

B. INTERDICTION MODELS

MCNIM is first described. It is a straightforward extension of MFNIM.

<u>Multi-Commodity Network-Interdiction Model (MCNIM)</u>

Indices:

$i, j \in N$	nodes in an undirected network $G = (N, A)$
$(i, j) \in A$	undirected arcs in the network $G = (N, A)$
N'_k	subset of "special nodes," $k = 1, \ldots, K$.
	$N_k \cap N_{k'} = \varnothing \quad \forall k \neq k'$
N'	$N' = \bigcup_{k=1}^{K} N'_k$

Data:

u_{ij} nominal capacity of arc (i, j)

r_{ij} interdiction resource required to interdict (break) arc (i, j)

R total interdiction resource

Decision Variables:

Network User's Decision Variables:

y_{ijk} amount of flow on arc (i, j) whose source is in N'_k

Interdictor's Decision Variables:

x_{ij} 1 if arc (i, j) is interdicted; 0 otherwise

Formulation:

$$\min_{\mathbf{x}\in X} \max_{\mathbf{y}} \sum_k \Big(\sum_{(i,j)\in A: i\in N'_k} y_{ijk} + \sum_{(i,j)\in A: j\in N'_k} y_{jik} \Big) \tag{31}$$

$$\text{s.t.} \quad \sum_{(i,j)\in A^+} y_{ijk} - \sum_{(j,i)\in A^+} y_{jik} = 0 \qquad k = 1,\ldots,K,\ i \in N - N' \tag{32}$$

$$\sum_k (y_{ijk} + y_{jik}) \le u_{ij}(1 - x_{ij}) \qquad (i, j) \in A \tag{33}$$

$$y_{ijk} \ge 0,\ y_{jik} \ge 0 \qquad k = 1,\ldots,K,\ (i, j) \in A$$

$$y_{ijk} \equiv 0 \qquad k = 1,\ldots,K,\ i \in N' - N'_k,\ (i, j) \in A \tag{34}$$

$$y_{jik} \equiv 0 \qquad k = 1,\ldots,K,\ j \in N' - N'_k,\ (i, j) \in A \tag{35}$$

$$y_{ijk} \equiv 0 \qquad k = 1,\ldots,K,\ j \in N'_k,\ (i, j) \in A \tag{36}$$

$$y_{jik} \equiv 0 \qquad k = 1,\ldots,K,\ i \in N'_k,\ (i, j) \in A \tag{37}$$

where $X = \{\mathbf{x} \in \{0,1\}^{|A|} : \sum_{(i,j)\in A} r_{ij} x_{ij} \le R\}$.

The objective (31) is to minimize the maximum flow among the subsets N'_k. The arc capacity constraints (33) restrict the amount of flow on each arc to the arc's nominal capacity if the arc is not interdicted, or to zero if the arc is interdicted.

For $\mathbf{x} \equiv \mathbf{0}$, the inner maximization is simply the "multi-commodity maximum-flow model" (MCMFM). MCMFM models the enemy's potential transfers of materiel among the subsets N'_k using K single-commodity flow models linked by joint capacity constraints. In MCMFM, for each subset N'_k there is a single-commodity maximum-flow model in which nodes in N'_k are treated as sources and nodes in $\bigcup_{k' \neq k} N'_{k'}$ are treated as sinks. Each single-commodity flow model is formulated using the approach of SMFM: Instead of defining a super-source connected to N'_k and a super-sink connected to $N' - N'_k$ and maximizing flow on a return arc, the amount of flow leaving N'_k is maximized; see the objective (31). Flow-balance constraints (32) are given only for non-special nodes; flows originating at sink nodes are fixed to 0 by (34) and (35), and flows entering source nodes are fixed to 0 by (36) and (37).

Like MFNIM, MCNIM is a difficult, non-convex minimization problem: In particular, the optimal solution to the inner maximization is a concave function of $\mathbf{x}$. However, the model can be convexified by moving the variables x_{ij} into the objective of the inner maximization as we did in the derivation of MFNIMc. We do this next.

Convexified Multi-Commodity Network-Interdiction Model (MCNIMc)

Indices:

As in MCNIM

Data:

As in MCNIM

Decision Variables:

As in MCNIM

Formulation:

$$\min_{\mathbf{x}\in X} \max_{\mathbf{y}} \sum_k \Big(\sum_{(i,j)\in A: i\in N'_k} y_{ijk} + \sum_{(i,j)\in A: j\in N'_k} y_{jik} \Big) - \sum_{(i,j)\in A} \Big(\sum_k (y_{ijk} + y_{jik}) \Big) x_{ij} \tag{38}$$

$$\text{s.t.} \sum_{(i,j)\in A^+} y_{ijk} - \sum_{(j,i)\in A^+} y_{jik} = 0 \qquad k = 1,\ldots,K,\ i \in N - N' \qquad : \alpha_{ik} \tag{39}$$

$$\sum_k (y_{ijk} + y_{jik}) \le u_{ij} \qquad (i,j) \in A \qquad : \beta_{ij} \tag{40}$$

$$y_{ijk} \ge 0,\ y_{jik} \ge 0 \qquad k = 1,\ldots,K,\ (i,j) \in A$$

$$y_{ijk} \equiv 0 \qquad k = 1,\ldots,K,\ i \in N' - N'_k,\ (i,j) \in A \tag{41}$$

$$y_{jik} \equiv 0 \qquad k = 1,\ldots,K,\ j \in N' - N'_k,\ (i,j) \in A \tag{42}$$

$$y_{ijk} \equiv 0 \qquad k = 1,\ldots,K,\ j \in N'_k,\ (i,j) \in A \tag{43}$$

$$y_{jik} \equiv 0 \qquad k = 1,\ldots,K,\ i \in N'_k,\ (i,j) \in A \tag{44}$$

where $X = \{\mathbf{x} \in \{0,1\}^{|A|} : \sum_{(i,j)\in A} r_{ij} x_{ij} \le R\}$. The α_{ik} and β_{ij} are dual variables for constraints (39) and (40), respectively, given fixed **x**.

The objective (38) is to minimize the total maximum flow less the flow on interdicted arcs. The inner objective in (38) is to maximize the maximum flow through

the network less flow on interdicted arcs. This is equivalent to maximizing flow with $\sum_k (y_{ijk} + y_{jik}) \le u_{ij}(1 - x_{ij}) \quad \forall\, (i,j) \in A$ as in the inner maximization of MCNIM because: As in the single-commodity model, 1 is an upper bound on the dual variable β_{ij} associated with the joint capacity constraints for arc (*i,j*) when $x_{ij} = 1$. This establishes that MCNIM and MCNIMc are essentially equivalent. Constraints (39) and (41) through (44) are as in MCNIM. Constraints (40) require that the total flow on an arc (*i,j*) not exceed that arc's capacity.

C. AN EQUIVALENT INTEGER PROGRAM

MCNIMc can be converted into a simple minimization model by fixing **x** temporarily, taking the dual of the inner maximization and then releasing **x**:

Multi-Commodity Network-Interdiction Model As An Integer Program (MCNIM-IP)

Indices:

As in MCNIM

Data:

As in MCNIM, plus

δ_{ijk} 1 if $i \in N'_k$; 0 otherwise

Decision Variables:

x_{ij} 1 if arc (i,j) is interdicted; 0 otherwise

α_{ik} dual variables associated with flow-balance constraints (39)

β_{ij} dual variables associated with capacity constraints (40)

Formulation:

$$\min_{\mathbf{x},\alpha,\beta} \quad \sum_{(i,j)\in A} u_{ij}\beta_{ij} \tag{45}$$

$$\text{s.t} \quad -\alpha_{ik} + \alpha_{jk} + \beta_{ij} + x_{ij} \geq \delta_{ijk} \qquad k = 1,\ldots,K,\ (i,j)\in A \tag{46}$$

$$-\alpha_{jk} + \alpha_{ik} + \beta_{ij} + x_{ij} \geq \delta_{jik} \qquad k = 1,\ldots,K,\ (i,j)\in A \tag{47}$$

$$\sum_{(i,j)\in A} r_{ij}x_{ij} \leq R \tag{48}$$

$$x_{ij} \in \{0,1\}, \beta_{ij} \geq 0 \qquad (i,j)\in A$$

$$\alpha_{ik} \text{ free} \qquad k = 1,\ldots,K,\ i\in N$$

The objective function (45) derives from the dual objective of MCMFM just as the objective of MFNIM derives from the dual of the standard maximum-flow model. However, we cannot guarantee that the α_{ik} or the β_{ij} here will be binary as they are in MFNIM; thus, there is no simple interpretation of this objective in terms of cut capacity as there is for the objective of MFNIM. Constraints (46) and (47) account for the arcs that (a) start at a non-special node and end at a sink node, (b) start at a source node and end at non-special node, (c) start at a non-special node and end at non-special node, and (d) start at a source node and end at sink node. Constraint (48) is the interdiction resource constraint, as usual.

D. COMPUTATIONAL RESULTS

We have tested MCNIM-IP using the test networks, computer, solver and optimality criterion that were used to test MPNIM. Table 3 gives model statistics and run times for the same values of R and K as in Table 1. We find optimal solutions to 10 out of 16 problems in a reasonable amount of time (at most 3,600 seconds). MCNIM-IP's

solution to the problem in Figure 1 is identical in **x** to the solution provided by MPNIM and objective values are equal. This must be true because we identify a feasible solution to MCNIM from the solution to MPNIM, and all arcs crossing the MPNIM partition are saturated in that solution.

In the solution just mentioned, all $\beta_{ij} = 0$ or 1. Those β_{ij} that are 1 correspond to the saturated, inter-subset arcs. Figure 2 shows one solution in which not all the β_{ij} are binary.

Network $G=(N,A)$	n_1	n_2	$\lvert N\rvert$	$\lvert A\rvert$	Total Resource R	Number of Subsets K	Initial Integrality gap (%)	Optimal Objective Value	Run Times (sec)
1	7	4	28	63	9	3	4.4	16.0	4.4
					11	3	100.0	0.0	2.9
					6	4	24.9	322.5	2.5
					11	4	63.9	144.0	2.6
2	10	6	60	149	10	3	39.8	321.0	6.3
					11	4	59.7	420.0	656.0
					16	4	90.2	221.0	408.0
					25	4	100.0	0.0	24.6
3	14	7	98	263	11	3	52.6	234.0	19.4
					20	3	100.0	0.0	39.3
					16	4	80.4	[313-2.4%]	3600.0
					20	4	100.0	[149-3.7%]	3600.0
4	14	9	126	333	11	4	44.5	[791-23.7%]	3600.0
					20	4	87.4	[476-75%]	3600.0
					11	5	35.1	[1102-25.9%]	3600.0
					20	5	64.4	[846-77%]	3600.0

Table 3. Model statistics and run times for MCNIM-IP using a relative optimality criterion of 1%. Several different values of R and K are used with four different networks. For instance, the problem for Network 3, which has 98 nodes and 263 arcs, when R=11 and K=3 is solved in 19.4 seconds with an optimal objective value of 234. The initial integrality gap for this problem is 52.6%. "[ ]" indicates that the problem could not be solved in less than 3,600 seconds with 1% OptCR. For those problems, the resulting objective value and OptCR at the end of 3,600 seconds are given in [ ]. For example, the problem for Network 3 when R=16 and K=4 has an objective value of 313 and an OptCR of 2.4% at the end of 3,600 seconds.

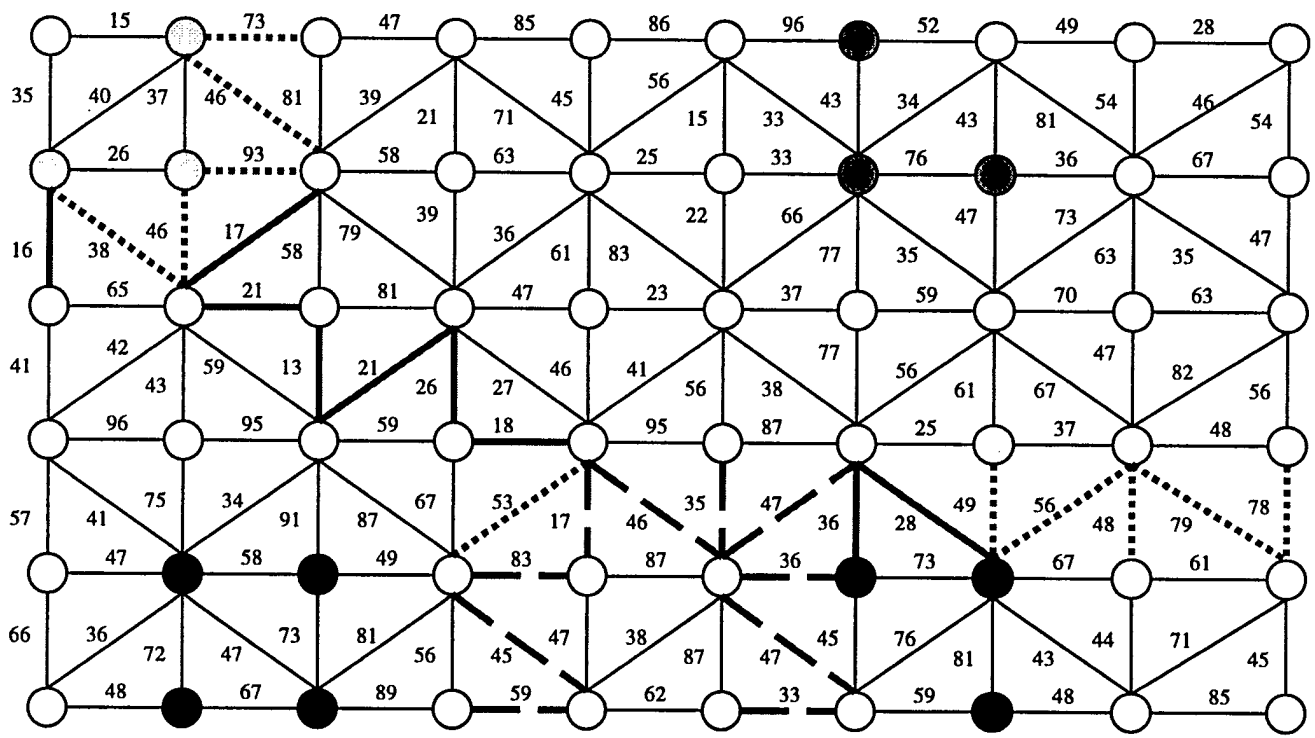
E. COMPARISON OF COMPUTATIONAL RESULTS

Table 4 compares solution statistics for MPNIM and MCNIM-IP. "By theorem," we know that $Z^*_{\text{MPNIM}} \geq Z^*_{\text{MCNIM-IP}}$, and the statistic $100\% \times Z^*_{\text{MPNIM}} / Z^*_{\text{MCNIM-IP}}$ allows for a

simple comparison of the two optimal objective values when both models can be solved. Unfortunately, MCNIM-IP can only be solved for 10 of the 16 problems. Of the 10 problems that are solved by both models, 7 have essentially identical solutions, and the largest difference is 3.6%. It appears that, at least for the problems solved by both models, there is a strong correlation between optimal objective values. The solution provided by MPNIM may be, in fact, a more-than-adequate approximation to the solution to MCNIM-IP, especially because MPNIM appears to be much easier to solve.

Network $G=(N,A)$	n_1	n_2	$\|N\|$	$\|A\|$	R	K	MPNIM		MCNIM-IP		$100\% \times \frac{Z^*_{MPNIM}}{Z^*_{MCNIM\text{-}IP}}$
							Z^*	Run times (sec)	Z^*	Run times (sec)	
1	7	4	28	63	9	3	16	1.6	16.0	4.4	100.0%
					11	3	0	1.8	0.0	2.9	100.0%
					6	4	334	1.7	322.5	2.5	103.6%
					11	4	144	1.8	144.0	2.6	100.0%
2	10	6	60	149	10	3	321	4.2	321.0	6.3	100.0%
					11	4	427	20.3	420.0	656.0	101.7%
					16	4	222	18.2	221.0	408.0	100.5%
					25	4	0	25.6	0.0	24.6	100.0%
3	14	7	98	263	11	3	234	4.2	234.0	19.4	100.0%
					20	3	0	39.5	0.0	39.3	100.0%
					16	4	313	6.4	[313-2.4%]	3600.0	-
					20	4	149	5.9	[149-3.7%]	3600.0	-
4	14	9	126	333	11	4	863	189.2	[791-23.7%]	3600.0	-
					20	4	426	160.9	[476-75%]	3600.0	-
					11	5	1414	518.5	[1102-25.9%]	3600.0	-
					20	5	924	563.1	[846-77%]	3600.0	-

Table 4. Model statistics and run times for MCNIM-IP and MPNIM using a relative optimality criterion of 1%. The problem for Network 2 when R=11 and K=4 is solved in 20.3 seconds with an objective value of 427 using MPNIM and in 656 seconds with an objective value of 420 using MCNIM-IP. The solutions differ by 1.7%. "[ ]" indicates that the model could not be solved in 3,600 seconds, and shows the best solution found and the integrality gap at termination.



Arcs that are interdicted, i.e., $x_{ij} = 1$. (There are 11 of these arcs.)

Arcs that are not interdicted and have $\beta_{ij} = 1$. (There are 9 of these arcs.)

Arcs that are not interdicted and have $\beta_{ij} = 0.5$. (There are 10 of these arcs.)

Arcs that are not interdicted and have $\beta_{ij} = 0$. (There are 119 of these arcs.)

Figure 2. Solution of MCNIM-IP to Network 2 when R=11, K=4 and $r_{ij} = 1 \ \forall \ (i, j) \in A$. This solution has some fractional values for β_{ij}. Numbers next to arcs are capacities. The optimal objective value is 420. Each shade of darkened node indicates a different subset N'_k. By linear programming duality, arcs with $\beta_{ij} > 0$ must be capacitated in the primal, post-interdiction flow model.

V. CONCLUSION

A. SUMMARY

This thesis has studied the K-group maximum-flow network-interdiction problem (KNIP) which is a generalization of the maximum-flow network-interdiction problem (MFNIP). In KNIP, a network user attempts to maximize flow among prespecified, disjoint "special node groups", $N_1', N_2', ..., N_K'$, $K \geq 3$, while an interdictor interdicts (destroys, stops flow on) network arcs, using limited interdiction resources, to minimize this maximum flow. We have developed two models, namely the multi-partition network-interdiction model (MPNIM) and the multi-commodity network-interdiction model (MCNIM) to solve KNIP, and showed how MPNIM can be modified to completely isolate the special node groups.

MPNIM is an approximating model that partitions the node set N into subsets $N_1, ..., N_K$, with $N_1' \subseteq N_1, ..., N_K' \subseteq N_K$, and interdicts certain arcs connecting the subsets N_k while observing constraints on interdiction resources. The objective is to minimize the total capacity of the uninterdicted arcs crossing between the various subsets.

MCNIM is an exact model that explicitly minimizes the maximum amount of flow that can potentially be moved among the node groups using K single-commodity flow models connected by joint capacity constraints. Viewed as a minimization, MCNIM is a difficult, non-convex problem. So we derive an equivalent convexified formulation, MCNIMc. Through MCNIMc, an equivalent integer program, MCNIM-IP, is obtained and solved.

We have tested MPNIM and MCNIM-IP using four artificially constructed test networks ranging in size from 28 nodes and 63 arcs to 126 nodes and 333 arcs. All computation was performed on a 333 MHz Pentium II computer using the CPLEX Version 6.5 solver. MPNIM, when used to isolate node groups, solves 8 problems (two values of K are used for each network) in less than 484.4 seconds each. When used to solve KNIP, MPNIM solves all 16 problems (two values of K and two resource levels for each network), each in less than 563.1 seconds. MCNIM-IP can only solve 10 of the problems given a limit of 3,600 CPU seconds. The optimal objective for MPNIM, Z^*_{MPNIM}, is an upper bound on the optimal objective for MCNIM-IP, $Z^*_{\text{MCNIM-IP}}$. For the 10 problems solved by both models, Z^*_{MPNIM} does not exceed $Z^*_{\text{MCNIM-IP}}$ by more than 3.6%, so MPNIM may be an adequate approximation and replacement for MCNIM in many situations.

B. FUTURE WORK

Further research on MCNIM and MPNIM is needed to improve solution times, enable solution of large problems, and broaden their scope of applications. To enable solution of large problems and shorten run times, solving MCNIMc by Benders decomposition can be tried. Cormican (1995) shows that Benders decomposition tends to work better than solving the integer program for MFNIP directly. To improve solvability (i.e., to enable solution of large problems and shorten run times) of both MPNIM and MCNIM, integer-programming cuts for both MPNIM and MCNIM-IP can be devised. Wood (1993) shows how such cuts can be beneficial in solving one maximum-flow interdiction model.

This thesis has ignored issues of uncertainty in the models, but uncertainty in intelligence reports and in interdiction success might be important. The models should be extended to handle uncertain interdiction successes and uncertain arc capacities as in Cormican, Morton and Wood (1996).

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